

# Renormalization-Scheme-Independent Perturbation Theory by Resumming Logarithms

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## Abstract

Results of perturbation theory in quantum field theory generally depend on the renormalization scheme that is in use. In particular, they depend on the scale. We try to make perturbation theory scheme invariant by re-expanding with respect to a scheme invariant quantity. Furthermore, we investigate whether the potentially large logarithms in such an expansion cause inaccuracy and how this can be improved.

## 1 Introduction

The occurrence of divergencies in perturbative quantum field theory has made it clear that although measurable quantities should be finite, this need not be the case for the parameters of the theory. To handle this, divergencies are regularized. To obtain an order-by-order finite perturbation expansion, it should be specified what to do with divergencies as they occur at each loop level. This introduces an arbitrary constant for each loop level. Although the full perturbation series should not depend on these constants, the truncated one does. This is a problem if one is trying to approximate a physical quantity that, by the definition of “physical quantity”, should not depend on these unphysical parameters.

It has been pointed out in [10] that scheme invariant quantities can be constructed out of the scheme-dependent ones. Therefore, the natural thing to do would be to try to rewrite the perturbation series into a series expansion with respect to the quantity  $X_1$ , the invariant that can be calculated at the one loop level. Actually, that would be a series expansion in  $1/X_1$ , since  $1/X_1$  is the small quantity if we are in the perturbative regime.

An expansion in  $1/X_1$  will, however, contain logarithms in the expansion coefficients. At high energies, these logarithms will dominate all other contributions to the expansion coefficients. We will resum these logarithms and investigate whether this gives more reliable results.

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Several attempts to remedy this problem of renormalization scheme dependence have been proposed. They all suffer from a great deal of arbitrariness and/or the mathematics involved is more complicated than that of simply manipulating power series, as we will be doing.

To obtain our results, we used the simple case of a one-parameter theory. We did not yet consider more complicated cases, but will do this later. When doing explicit calculations, we use QCD with five massless quarks, as a concrete example of a one-parameter theory.

## 2 Renormalization Scheme Invariants

In this section we outline how renormalization scheme-independent quantities can be combined into scheme invariant ones. This is also explained in [10] and [6]. For self-containedness of this paper and also to make clear what our conventions are, we will repeat this in this section. The coupling constant will be denoted by  $a$ . The key idea is that the consistency of perturbation theory requires that a result up to terms of order  $a^n$  should only differ from the exact answer up to terms of order  $a^{n+1}$ . If we consider the case  $n = 1$ , we see that the running of the coupling makes the use of  $a$  as an expansion parameter scheme dependent. This indeed is an effect that starts at order  $a^2$ . I.e., the lowest order term in the beta function is of order  $a^2$ .

The renormalization scheme can be specified by giving the renormalization scale  $s$  and the scheme-dependent beta function coefficients  $\beta_2, \beta_3, \beta_4, \dots$ . The coupling constant depends on the scale via

$$\frac{\partial a}{\partial s} = \beta(a) = \beta_0 a^2 + \beta_1 a^3 + \beta_2 a^4 + \dots, \quad (1)$$

where  $ds = d\mu/\mu$  with  $\mu$  the mass scale used in dimensional regularization. This differential equation can be integrated to give

$$s = -\frac{1}{\beta_0 a} + \frac{\beta_1}{\beta_0^2} \log \frac{\beta_0 + \beta_1 a}{\beta_1 a} + \int_0^a da' \left( \frac{1}{\beta(a')} - \frac{1}{\beta_0 (a')^2 + \beta_1 (a')^3} \right). \quad (2)$$

This solution implies the choice of a boundary value. Our choice is similar to the one in [10]. Taking the set of variables that consists of  $s$  and the scheme-dependent beta function coefficients as independent, we can derive that

$$\frac{\partial a}{\partial \beta_i} = \beta(a) \int_0^a da' \frac{(a')^{i+2}}{\beta(a')^2}. \quad (3)$$

As an example, we imagine that we have calculated a physical quantity  $R$  up to fourth order. From this example, we hope, it will be clear how this can be generalized to arbitrary order. Having calculated  $R$  up to fourth order means that we have

$$R \sim r_0 a + r_1 a^2 + r_2 a^3 + r_3 a^4. \quad (4)$$

Consistency of perturbation theory requires that

$$\frac{dR}{d \text{scheme}} \sim a^5. \quad (5)$$

To be more concrete, independence of  $s$  implies that

$$(r_0 + 2r_1a + 3r_2a^2 + 4r_3a^3) \frac{\partial a}{\partial s} + \frac{\partial r_1}{\partial s}a^2 + \frac{\partial r_2}{\partial s}a^3 + \frac{\partial r_3}{\partial s}a^4 \sim a^5. \quad (6)$$

From this, equations for second, third and fourth order can be extracted. Substituting the beta function for  $\partial a/\partial s$ , we find

$$\begin{aligned} r_0\beta_0 + \frac{\partial r_1}{\partial s} &= 0; \\ 2\beta_0r_1 + r_0\beta_1 + \frac{\partial r_2}{\partial s} &= 0; \\ 3\beta_0r_2 + 2\beta_1r_1 + r_0\beta_2 + \frac{\partial r_3}{\partial s} &= 0. \end{aligned} \quad (7)$$

To obtain the equations that follow from independence of  $\beta_2$ , we expand  $\partial a/\partial \beta_2$  up to fourth order. We have

$$\frac{\partial a}{\partial \beta_2} = \frac{1}{\beta_0}a^3 + O(a^5). \quad (8)$$

Using this and then demanding that the third and fourth order of  $\partial R/\partial \beta_2$  are zero, we get the equations

$$\begin{aligned} \frac{r_0}{\beta_0} + \frac{\partial r_2}{\partial \beta_2} &= 0; \\ 2\frac{r_1}{\beta_0} + \frac{\partial r_3}{\partial \beta_2} &= 0. \end{aligned} \quad (9)$$

Finally, independence of  $\beta_3$  gives the equation

$$\frac{r_0}{2\beta_0} + \frac{\partial r_3}{\partial \beta_3} = 0. \quad (10)$$

Integrating the equations for  $r_1$ ,  $r_2$ , and  $r_3$ , we find

$$\begin{aligned} r_1 &= X_1 - \beta_0 r_0 s; \\ r_2 &= X_2 + \frac{r_1^2}{r_0} - \frac{r_0 \beta_2}{\beta_0} + \frac{\beta_1 r_1}{\beta_0}; \\ r_3 &= X_3 + \frac{3r_1 X_2}{r_0} + \frac{5\beta_1 r_1^2}{2\beta_0 r_0} - \frac{r_0 \beta_3}{2\beta_0} - \frac{2r_1 \beta_2}{\beta_0} + \frac{r_1^3}{r_0^2}, \end{aligned} \quad (11)$$

where the quantities  $X_i$  are the renormalization scheme invariants. They arise because the values of the  $r_i$  are obtained from differential equations and hence need constants of integration. The values of the  $X_i$  can be calculated from the values of  $r_i$  and  $\beta_i$  once these have been calculated. It should be made sure that the  $r_i$  and  $\beta_i$  have been calculated in the same renormalization scheme, of course, and then it will turn out that the values of  $X_i$  no longer depend on the scheme that was used to obtain  $r_i$  and  $\beta_i$ .

The acute reader will have noticed that we actually expressed  $r_2$  and  $r_3$  into  $r_1$  instead of in  $s$ . Hence, we actually label our renormalization scheme by  $r_1, \beta_2, \beta_3, \dots$  rather than by the set of parameters that we mentioned earlier. The reason that we do this is that if we are going to re-expand into  $1/X_1$  we would rather have  $r_1$  than  $s$  in the expansion coefficients because in a suitable

renormalization scheme  $r_1$  will be a number of order unity while  $s$  will be of order  $1/a$ , which is large. In the end it will hopefully turn out that the result will not depend on what kind of “suitable renormalization scheme” we used, however, we are not yet at this point so we should still make sure that we do not have large expansion coefficients.

### 3 Expansion in $1/X_1$

For an expansion to work, we need to know that we are expanding with respect to a small parameter. The idea of perturbation theory is that the coupling constant,  $a$ , is a small quantity. Equation (2) can be expanded in powers of  $a$ . We have

$$s \sim -\frac{1}{\beta_0 a} + \frac{\beta_1}{\beta_0^2} \log\left(\frac{\beta_0}{\beta_1 a}\right) + \frac{\beta_1^2 a}{\beta_0^3} - \frac{\beta_2 a}{\beta_0^2} - \frac{\beta_1^3 a^2}{2\beta_0^4} - \frac{\beta_3 a^2}{2\beta_0^2} + \frac{\beta_1 \beta_2 a^2}{\beta_0^3} - \frac{\beta_4 a^3}{3\beta_0^2} + \frac{\beta_2^2 a^3}{3\beta_0^3} - \frac{\beta_2 \beta_1^2 a^3}{\beta_0^4} + \frac{2\beta_3 \beta_1 a^3}{3\beta_0^3} + \frac{\beta_1^4 a^3}{3\beta_0^5}. \quad (12)$$

From this series we see that if  $a$  is small,  $s$  must be large. Indeed, in the case of QCD,  $s$  is given by

$$s = \log\left(\frac{\mu^2}{\Lambda_{\text{QCD}}^2}\right), \quad (13)$$

where  $\mu$  is the renormalization scale as it occurs in dimensional regularization and  $\Lambda_{\text{QCD}}$  is of the order of the energy range where the coupling is large and perturbation theory is not accurate. Therefore, the perturbation series can also be written as an expansion in  $1/s$ . For this we use the inverse of the expansion 12. We have

$$a \sim -\frac{1}{\beta_0 s} + \frac{\beta_1}{\beta_0^3 s^2} L_s + \frac{\beta_1^2}{\beta_0^5 s^3} - \frac{\beta_2}{\beta_0^4 s^3} - \frac{\beta_1^2}{\beta_0^5 s^3} L_s - \frac{\beta_1^2}{\beta_0^5 s^3} L_s^2 - \frac{\beta_1^3}{2\beta_0^7 s^4} + \frac{\beta_3}{2\beta_0^5 s^4} - \frac{2\beta_1^3}{\beta_0^7 s^4} L_s + \frac{3\beta_2 \beta_1}{\beta_0^6 s^4} L_s + \frac{\beta_1^3}{\beta_0^6 s^4} L_s^3 + \frac{5\beta_1^3}{2\beta_0^7 s^4} L_s^2, \quad (14)$$

where  $L_s = \log(-\beta_1/(\beta_0^2 s))$ .

Now we assume that a suitable renormalization scheme has been chosen, and hope to obtain results that turn out not to depend on our “suitable renormalization scheme” and hence might also have been calculated if we had started out with an unsuitable renormalization scheme. In a suitable renormalization scheme, we expect that expansion coefficients are not large. In particular  $r_1$  is expected to be of order unity. From equation (11) it follows that  $X_1 = r_1 + \beta_0 r_0 s$ . We conclude that  $X_1$  must be a large quantity, because  $s$  is. Hence, the expansion

$$\frac{1}{s} = \frac{\beta_0 r_0}{X_1 - r_1} \sim \frac{\beta_0 r_0}{X_1} + \frac{\beta_0 r_0 r_1}{X_1^2} + \frac{\beta_0 r_0 r_1^2}{X_1^3} + \frac{\beta_0 r_0 r_1^3}{X_1^4} \quad (15)$$

is a good expansion. We substitute this equation into equation (14) obtaining an expansion of  $a$  in  $1/X_1$ . This expansion is substituted into equation (4). We then obtain an expansion of the physical quantity  $R$  into  $1/X_1$ . Furthermore, in this expansion, we substitute for  $r_2, r_3, r_4, \dots$  the values as given by equation (11).

It should be noted that although  $X_1$  does not depend on the renormalization scheme, it is dependent upon the physical quantity under consideration. There is, however, nothing wrong with using a different expansion parameter for every different physical quantity. The expansion that is obtained by making all these substitutions is

$$R \sim -\frac{r_0^2}{X_1} + \frac{r_0^3\beta_1}{\beta_0 X_1^2} L_X - \frac{r_0^3 X_2}{X_1^3} + \frac{r_0^4\beta_1^2}{\beta_0^2 X_1^3} (1 - L_X - L_X^2) \\ + \frac{r_0^4 X_3}{X_1^4} + \frac{3r_0^4\beta_1 X_2}{\beta_0 X_1^4} L_X + \frac{r_0^5\beta_1^3}{\beta_0^3 X_1^4} \left(-\frac{1}{2} - 2L_X + \frac{5}{2}L_X^2 + L_X^3\right), \quad (16)$$

where  $L_X$  is given by  $L_X = \log(-r_0\beta_1/(\beta_0 X_1))$ . We see that in this expansion all scheme-dependent terms have canceled. These scheme-dependent terms are the ones involving  $r_1$  and  $\beta_i$  with  $i \geq 2$ . Because of the cancellation of these terms, we have obtained a scheme-independent perturbation series.

For this method to work for any order in perturbation theory, we must prove that the cancellation of scheme-dependent terms happens at every order and not just up to fourth order. For this purpose, note that our expansion is an expansion with respect to the variables  $1/X_1, 1/\beta_0, \beta_1, \beta_2, \beta_3, \dots, r_0, r_1, X_2, X_3, X_4, \dots, L_X$ . If we refer to the order of a term in the series, we mean the order in  $1/X_1$ . We should prove that actually the variables  $r_1, \beta_2, \beta_3, \dots$  do not occur in this series expansion. Let us assume that such a variable actually does occur at some order  $n$  in the expansion.

We introduce  $v$  as an alias for one of the offending variables (there might be several offending variables) that occurs at order  $n$ . This means that  $\partial R/\partial v$  is of order  $n$ . We consider what happens if we re-expand the expression for  $R$  in  $a$  again, and then differentiate with respect to  $v$ . To do this we first have to expand  $1/X_1 = 1/(\beta_0 r_0 s + r_1)$  in  $1/s$  and then use equation (12) to expand this in  $a$  again.  $X_1$  was defined in such a way that it actually does not depend on scheme-dependent quantities such as  $v$ . Therefore, we know that the entire series of  $1/X_1$  in  $a$  does not depend on  $v$ . Furthermore, we note that during re-expanding and differentiating the order of a term in  $1/X_1, 1/s$  or  $a$ , whichever applies, never decreases. Therefore, all the invariant terms up to order  $n$  after differentiation cause terms that are at least of order  $n+1$ . The only terms that can give a contribution of order  $n$  are the scheme-dependent terms. However, the terms obtained by differentiating  $a$ , using the chain rule, are of higher order, so this does not contribute. The conclusion is that the derivative with respect to  $v$  up to order  $n$  is the same before re-expanding as after re-expanding, except that  $1/X_1$  is to be replaced by  $-a/r_0$  and  $L_X$  is to be replaced by  $\log(\beta_1 a/\beta_0)$ . Hence, the physical quantity  $R$  up to order  $n$  depends on the scheme at order  $n$ . This is a contradiction with the starting point of section 2. Therefore the expansion in  $1/X_1$  must have renormalization scheme independent coefficients.

## 4 Resumming the Logarithms

In this section, we will resum all logarithms  $L_s$  that occur in equation (14). We start out by observations that have been made from computer algebra experimentation, but in the end we will prove our results to be correct to all orders.

By looking at the expansion (14), we observe that the leading logarithms, i.e., terms of the order  $L_s^{n-1}/s^n$  can be summed into the quantity  $1/s'$  defined by

$$\frac{1}{s'} = \frac{1}{s} \frac{1}{1 + \beta_1 L / (\beta_0^2 s)}. \quad (17)$$

After this resummation we have the expansion

$$\begin{aligned} a(s', L) \sim & -\frac{1}{\beta_0 s'} - \frac{\beta_2}{\beta_0^4 (s')^3} + \frac{\beta_1^2}{\beta_0^5 (s')^3} (1 - L) + \frac{\beta_1^3}{\beta_0^7 (s')^4} \left( -\frac{1}{2} + L - \frac{L^2}{2} \right) \\ & + \frac{\beta_3}{2\beta_0^5 (s')^4} + \frac{\beta_1^4}{\beta_0^9 (s')^5} \left( -\frac{7}{6} + 2L - \frac{1}{2}L^2 - \frac{1}{3}L^3 \right) \\ & + \frac{3\beta_1^2 \beta_2}{\beta_0^8 (s')^5} (1 - L) - \frac{5\beta_2^2}{3\beta_0^7 (s')^5} + \frac{\beta_1 \beta_3}{6\beta_0^7 (s')^5} - \frac{\beta_4}{3\beta_0^6 (s')^5} \end{aligned} \quad (18)$$

Looking at the highest order logarithms in this expansion, we recognize the expansion of  $\beta_1/(\beta_0^3 s^2) \log(1 - \beta_1 L/(\beta_0^2 s'))$ . We therefore define a quantity  $L'$  by

$$L' = \log \left( 1 - \frac{\beta_1}{\beta_0^2 s'} L \right). \quad (19)$$

It now turns out that after rewriting the expansion for  $a$  with respect to  $1/s'$  and  $L'$ , we recover the original expansion where  $1/s$  has been replaced by  $1/s'$  and  $L$  by  $L'$ . Because of this we can iterate this procedure and obtain a sequence of values  $1/s_n$  and  $L_n$  from the iteration

$$\begin{aligned} \frac{1}{s_{n+1}} &= \frac{1}{s_n} \frac{1}{1 + \frac{\beta_1}{\beta_0^2 s_n} L_n}; \\ L_{n+1} &= \log \frac{1}{1 + \frac{\beta_1}{\beta_0^2 s_n} L_n}, \end{aligned} \quad (20)$$

where we have rewritten  $L_{n+1}$  in terms of  $s_n$  and  $L_n$  instead of in terms of  $s_{n+1}$  and  $L_n$ . We note that this iteration increases the order of  $L$  with respect to  $1/s$ . Therefore, in the perturbative regime, the iteration should make  $L$  converge to zero. This resums all logarithms into a new value for  $1/s$ .

A curve can be drawn through the sequence of points  $(1/s_n, L_n)$ . This curve is given by

$$\begin{aligned} \frac{1}{s(x)} &= \frac{1}{s_\infty} \frac{1}{1 + \frac{\beta_1}{\beta_0^2 s_\infty} x}; \\ L(x) &= \log \frac{1}{1 + \frac{\beta_1}{\beta_0^2 s_\infty} x} - x, \end{aligned} \quad (21)$$

where  $x$  parameterizes the curve and different curves (for different initial values of  $1/s$  and  $L_s$ ) are labeled by  $1/s_\infty$ . That this is correct, can be seen by checking that the iteration for the pair  $(1/s, L)$  is recovered if  $x$  is iterated using the prescription

$$x_{n+1} = \log \frac{1}{1 + \frac{\beta_1}{\beta_0^2 s_\infty} x_n}. \quad (22)$$

We see that also for the  $x_n$  the property holds that  $x_{n+1}$  is of higher order than  $x_n$ , hence in the perturbative regime it should converge to zero. In this case

we note that  $1/s(x) \rightarrow 1/s_\infty$ , which explains our notation “ $1/s_\infty$ ” to label the different curves.

We must still prove our assertion that the iteration preserves the expansion to all orders in  $1/s$ . It is sufficient to show that the value of  $a$  is constant along the curves introduced above. The infinitesimal form of the curves is

$$\begin{aligned}\delta s &= \frac{\beta_1}{\beta_0^2} \delta x; \\ \delta L_s &= \left( -1 - \frac{\beta_1}{\beta_0^2 s} \right) \delta x\end{aligned}\tag{23}$$

Proving that this is a symmetry of the expansion (14) is also sufficient to see that the iteration works. This is what we will do in the next section.

## 5 Proof of the Iteration

Here we will prove that the transformation (23) is a symmetry of the expansion equation (14). This, at the same time, shows that the iteration of equation (20) works to all orders and that the logarithms in the expansion can be made zero. We introduce a quantity  $\bar{a}$  that is a power series in  $1/s$  and  $L_s$ . This quantity has the definition

$$\begin{aligned}\frac{\partial \bar{a}}{\partial(1/s)} &= - \left( s^2 + \frac{\beta_1 s}{\beta_0^2} \right) \beta(\bar{a}); \\ \frac{\partial \bar{a}}{\partial L_s} &= \frac{\beta_1}{\beta_0^2} \beta(\bar{a}); \\ [s\bar{a}(1/s, L_s = 0)]_{s \rightarrow \infty} &= -1/\beta_0.\end{aligned}\tag{24}$$

We will show that this quantity  $\bar{a}$  is actually the same as  $a$ . First note that these differential equations are consistent because  $\partial/(\partial(1/s))$  and  $\partial/\partial L_s$  commute. Secondly, if we confine ourselves to the surface  $L_s = \log(-\beta_1/(\beta_0^2 s))$  we have, as is verifiable by using the chain rule for differentiation,  $d\bar{a}/ds = \beta(\bar{a})$ , so, also using the boundary condition, we see that on this surface  $a$  and  $\bar{a}$  are the same. We now show that for every order in  $1/s$ , there is a finite number of terms. This ensures that also away from the surface  $L_s = \log(-\beta_1/(\beta_0^2 s))$  these functions are the same, because it is impossible to express  $s$  and  $L_s$  into each others in a finite number of terms. The pre-factors of the expansion in  $1/s$  and  $L_s$  of  $\bar{a}$  can be obtained by setting  $\bar{a}s = -1/\beta_0$  in the derivatives  $\partial^{n+m} \bar{a} / (\partial L_s^m \partial(1/s)^n)$ . At first sight, it may seem that problems could be caused by terms of the form  $\bar{a}^m s^n$  where  $n > m$ . However, the fact that the differential equations have a solution that is an expansion in  $L_s^m / s^n$  with  $n > 0$  and  $m \geq 0$ , ensures that these problematic contributions will cancel if we substitute  $\bar{a}s \rightarrow -1/\beta_0$ . Still, before this substitution is made, there will be terms of the form  $\bar{a}^p (1/s)^{p+q}$ , with  $p > 0$  and  $q \geq 0$ . If we consider the quantity  $d^n \bar{a} / d(1/s)^n$ , the maximum value of  $q$  for which this type of monomial will occur is equal to  $n-1$ . Differentiations with respect to  $L_s$  increase the order in  $a$ , hence, in  $\partial^{n+m} \bar{a} / (\partial L_s^m \partial(1/s)^n)$ , the maximum value for  $q$  will be  $n-1-m$ . Hence, if  $m \geq n$  we will only have terms containing  $\bar{a}^p (1/s)^q$  with  $p > q$ . If we substitute  $\bar{a}s = -1/\beta_0$ , taking  $s \rightarrow \infty$ , these terms will become zero and thus do not contribute. From this we see that the maximum order in  $L_s$  that occurs in the coefficient of  $1/s^p$  in the

expansion in  $s$  is  $p$ . Hence, this coefficient of  $1/s^p$  contains a finite number of terms, as we set out to show. We conclude that the quantity  $\bar{a}$  indeed has the same expansion with respect to  $1/s$  and  $L_s$  as  $a$ .

Furthermore, from equations (23) and (24), it follows that  $da/dx = 0$ . Therefore the symmetry (23) is indeed a symmetry of the expansion of  $a$  in  $1/s$  and  $L_s$ , and the iteration in equation (20) keeps the value of  $a$  constant to all orders in  $1/s$ .

## 6 Resumming $L_X$

The symmetry of equations (23) can be turned into a symmetry of the expansion of a physical quantity in  $1/X_1$  and  $L_X$ . This will enable us to perform resummation of logarithms in such an expansion. From  $X_1 = r_0\beta_0s + r_1$ , it follows that

$$\delta X_1 = r_0\beta_0(\delta s) = \frac{r_0\beta_1}{\beta_0}\delta x. \quad (25)$$

Turning  $L_s$  into  $L_X$  is done via

$$L_s = L_X + \log \frac{X_1}{X_1 - r_1}. \quad (26)$$

Applying  $\delta$  on both sides gives

$$\left(-1 - \frac{\beta_1}{\beta_0s}\right)\delta x = \delta L_X - \frac{r_1}{X_1}\delta X_1. \quad (27)$$

Equation 25 then gives

$$\delta L_X = -\left(1 + \frac{r_0\beta_1}{\beta_0X_1}\right)\delta x. \quad (28)$$

Using this symmetry it is possible to turn  $L_X$  into zero, thereby ridding ourselves of logarithms. The value of  $X_1$  that is obtained while turning  $L_X$  to zero will be called  $\tilde{X}_1$ . Integrating equations (25) and (28), we obtain the equation

$$X_1 = \tilde{X}_1 - \frac{r_0\beta_1}{\beta_0} \log \frac{-r_0\beta_1}{\beta_0\tilde{X}_1}. \quad (29)$$

This can be expressed in the Lambert  $W$ -function. This function is by definition the solution to  $W(z)e^{W(z)} = z$ . We have

$$\tilde{X}_1 = \frac{r_0\beta_1}{\beta_0} W\left(-e^{\frac{\beta_0X_1}{r_0\beta_1}}\right). \quad (30)$$

Hence, the conclusion is that we turned the standard perturbation theory into an expansion in the quantity  $1/\tilde{X}_1$ . The expansion looks as displayed in equation (16) with  $X_1$  replaced by  $\tilde{X}_1$  and all terms that have an  $L_X$  removed. Since this reduces the number of terms considerably, let us display a few more. We have

$$\begin{aligned} R \sim & -\frac{r_0^2}{\tilde{X}_1} + \frac{r_0^4\beta_1^2}{\beta_0^2\tilde{X}_1^3} - \frac{r_0^3X_2}{\tilde{X}_1^3} - \frac{r_0^5\beta_1^3}{2\beta_0^3\tilde{X}_1^4} + \frac{r_0^4X_3}{\tilde{X}_1^4} - \frac{7r_0^6\beta_1^4}{6\beta_0^4\tilde{X}_1^5} + \frac{3r_0^5\beta_1^2X_2}{\beta_0^2\tilde{X}_1^5} - \frac{r_0^5X_4}{\tilde{X}_1^5} \\ & + \frac{17r_0^7\beta_1^5}{12\beta_0^5\tilde{X}_1^6} - \frac{3r_0^6\beta_1^3X_2}{2\beta_0^3\tilde{X}_1^6} - \frac{4r_0^6\beta_1^2X_3}{\beta_0^2\tilde{X}_1^6} + \frac{r_0^6X_5}{\tilde{X}_1^6}. \end{aligned} \quad (31)$$



highest order	not using symmetry	using symmetry
$1/s$	91.5	91.5
$1/s^2$	282.2	249.0
$1/s^3$	247.5	249.3
$1/s^4$	248.6	249.0

Table 1: Values for  $\Lambda_{\text{QCD}}$  in MeV obtained by solving equation (12) to various orders numerically, while using or not using the symmetry in equation (23). We used a number of quarks equal to five to obtain this result.

## 7 Determining $\Lambda_{\text{QCD}}$

In their *Review of Particle Physics* [8] the Particle Data Group suggests using equation (14) to define  $\Lambda_{\text{QCD}}$ . To be fully accurate, their definition is not completely the same. Instead of our  $L_s = \log(-\beta_1/(\beta_0^2 s))$  the PDG uses  $L_s = -\log s$ . This amounts to a shift in the parameter  $s$ . A way to see this is from equation (2). Adding a factor  $-\beta_1/\beta_0^2$  inside the logarithm in this equation, turns our expansion into the one of the PDG. This is equivalent to adding a constant to the right-hand side of this equation. This constant can then be moved to the left-hand side, so we see that  $s$  is indeed shifted. The consequence is that the  $\Lambda_{\text{QCD}}$  that we use differs by a multiplicative constant from the one of the PDG. We have

$$\Lambda_{\text{QCD}}^{\text{PDG}} = \Lambda_{\text{QCD}}^{\text{OURS}} \left( -\frac{\beta_0^2}{\beta_1} \right)^{\beta_1/(2\beta_0^2)}, \quad (32)$$

where the beta function coefficients are given in our conventions. In the rest of the paper we use our conventions, hence we will be writing  $\Lambda_{\text{QCD}}$  for  $\Lambda_{\text{QCD}}^{\text{OURS}}$  and never mention  $\Lambda_{\text{QCD}}^{\text{PDG}}$  again. Note however, that the expansion with respect to  $1/X_1$  and  $L_X$  becomes different if we choose a different prefactor inside the logarithm. We will see in the next section that using the symmetry of equations (25) and (28) resolves this ambiguity.

The suggestion of the PDG to use equation (14) to define  $\Lambda_{\text{QCD}}$  is not very practical. In the first place it would seem to be easier to use equation (2). Secondly, if one is going to use equation (14), the symmetry from equation (23) is useful to obtain a series that converges faster by transforming  $L_s$  to zero. The PDG gives  $\alpha_s(M_Z) \approx 0.1187$ . In table 1 we compare the value obtained for  $\Lambda_{\text{QCD}}$  using the symmetry and not using it. The used beta function coefficients were calculated in [9] and recently confirmed in [2]. The number of flavours was set to five. We indeed see faster convergence.

Below we will be using  $\Lambda_{\text{QCD}}$ . The value that we use comes from equation (14) to fourth order where we use our symmetry to get rid of the  $L_s$ 's. If one uses standard perturbation theory, one has to pick a suitable value for  $\mu$  and specify the renormalization scheme. In that case, the consistent way to proceed is to only use the beta function up to the loop level to which the rest of the calculation is done. The difference with our case is that while in the standard approach  $\alpha_s(M_Z) \approx 0.1187$  appears as a fundamental constant, in our approach  $\Lambda_{\text{QCD}}$  would be the fundamental quantity. Determining the value of a fundamental constant from another one is better done with as much accuracy as possible. This is the reason that we use the beta function up to four loop level.

## 8 How Invariant is Invariant, Really?

Our method attempts to remedy the arbitrariness in choosing the renormalization prescription, so we should now ask the question to what extent this method itself is arbitrary. A first possible source of arbitrariness is the choice of variables to parameterize the renormalization prescription. We choose the set of parameters  $r_1, \beta_2, \beta_3, \beta_4, \dots$  to parameterize the renormalization prescription. If one chooses a different set of variables,  $\bar{r}_1 = r_1 + \Delta_1, \bar{\beta}_2 = \beta_2 + \Delta_2, \bar{\beta}_3 = \beta_3 + \Delta_3$ , where the  $\Delta$ 's are constants, the integration constants  $X_i$  from section 2 also become different. However, if we have  $\Delta_1 = 0$ , it turns out that in the end the final expansion coefficients still have the same value, so this is not an arbitrariness of our method.

This shows that the small invariant quantity that we decide to use for expansion is more important than the precise definition of the other invariants. The question that might arise is what would happen if we would expand with respect to some arbitrary function of  $X_1$  instead of with respect to  $X_1$ . We could, for instance, expand with respect to the sine of  $X_1$ . The possibility of expressing the coupling constant in one scheme as a power series in the coupling constant in another scheme is a possibility that has been mentioned in literature, for instance in [10]. Our point of view is that the possibility to use an arbitrary power series is not a fundamental arbitrariness of perturbation theory. Note that if it were, it would apply to any perturbative method in any branch of physics. Small quantities that are used for expansion should be the ones that come naturally with the problem under consideration, not the ones that can be used to show that any approximation method can be made to give wrong answers. In field theory, the situation is that no renormalization scheme is a priori better than any other, and this ambiguity can be parameterized by using an arbitrary power series in the coupling constant. This is the reason arbitrary power series of the coupling constant can be useful to consider. Considering the fact that the first scheme invariant arises naturally from the demand that a physical quantity should not depend on the scale, it does not make sense to consider arbitrary functions of this, perhaps with the exception of a translation in the definition of  $X_1$ . I.e., a non-zero value of  $\Delta_1$ .

If  $\Delta_1 \neq 0$ , we obtain a different invariant variable on the one loop level, namely  $X'_1 = X_1 + \Delta_1$ . If we were to expand in  $1/X'_1$ , we would indeed obtain a different expansion. Here, it appears, we have finally found arbitrariness. However, we actually already mentioned this. An example of this arbitrariness is given in the paragraph that contains equation (32). The prefactor that we choose inside the logarithm  $L_s$  is equivalent to choosing a value for  $\Delta_1$ . Resumming the logarithms yields the same result for  $\tilde{X}_1$  and, hence, also the same result for the physical quantity  $R$ .

## 9 Hadronic- $R$

In this section we consider massless QCD-corrections to hadronic  $R_{\text{had}}$ . This quantity is by definition given by

$$R_{\text{had}} = \frac{\sigma(e^+ + e^- \rightarrow \text{hadrons})}{\sigma(e^+ + e^- \rightarrow \mu^+ + \mu^-)}, \quad (33)$$

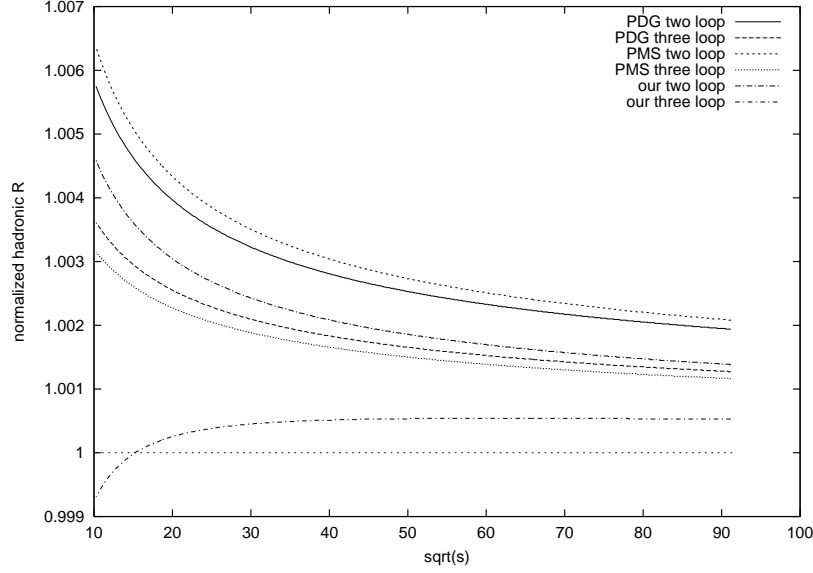


Figure 1: hadronic  $R$  as a function of the center of mass energy for  $N_f = 5$ , as found by various methods, normalized by dividing by the PDG-one-loop result. Note that our method gives rather different results from the PDG-method and the PMS-method. From top to bottom we see graphs for PMS-two-loop, PDG-two-loop, [gap], our-two-loop, PDG-three-loop, PMS-three-loop, [gap], and our-three-loop.

where electroweak corrections are neglected. The value of this quantity can be obtained up to third order from the review paper [1]. The beta function coefficients can be obtained from the same paper or from [9].  $a$  is taken to be  $\alpha_s/\pi$ . For the first renormalization scheme invariant  $X_1$ , we have

$$\begin{aligned}
 X_1 &= r_1 + \beta_0 r_0 s \\
 &= \frac{365}{24} - 11\zeta(3) - N_f \left( \frac{11}{12} - \frac{2}{3}\zeta(3) \right) + \left( -\frac{11}{4} + \frac{1}{6}N_f \right) \log \left( \frac{s_{\text{CM}}}{\Lambda_{\text{QCD}}^2} \right),
 \end{aligned}
 \tag{34}$$

where we have written  $s_{\text{CM}}$  for the squared center of mass energy, to avoid confusion with the quantity  $\log(\mu^2/\Lambda_{\text{QCD}}^2)$ , that we also call  $s$ . The addition of  $\beta_0 r_0 s$  has resulted in the replacement  $\mu \rightarrow \Lambda_{\text{QCD}}$ , in  $r_1$ . Note that the value of  $\Lambda_{\text{QCD}}$  is scheme dependent. For instance, if we would use the  $\overline{\text{MS}}$ -scheme (we are actually using  $\overline{\text{MS}}$ ), the two values of  $\Lambda_{\text{QCD}}$  would differ by a multiplicative constant. The scheme invariant is independent of this choice. A practical way to proceed is to start from the expansion of  $R_{\text{had}}$  up to and including terms of order  $a^3$ , substitute, equation (14) into it and expand up to terms of order  $1/s^3$ . In this we substitute equation (15) and expand again with respect to  $1/X_1$  up to terms of order  $1/X_1^3$ . Using computer algebra this is a more or less trivial thing to do. This cancels all dependence on the scheme, as we have argued. In particular, we observe that results no longer depend on the renormalization mass  $\mu$ .

We plot a normalized variant of hadronic  $R$  as calculated by two other

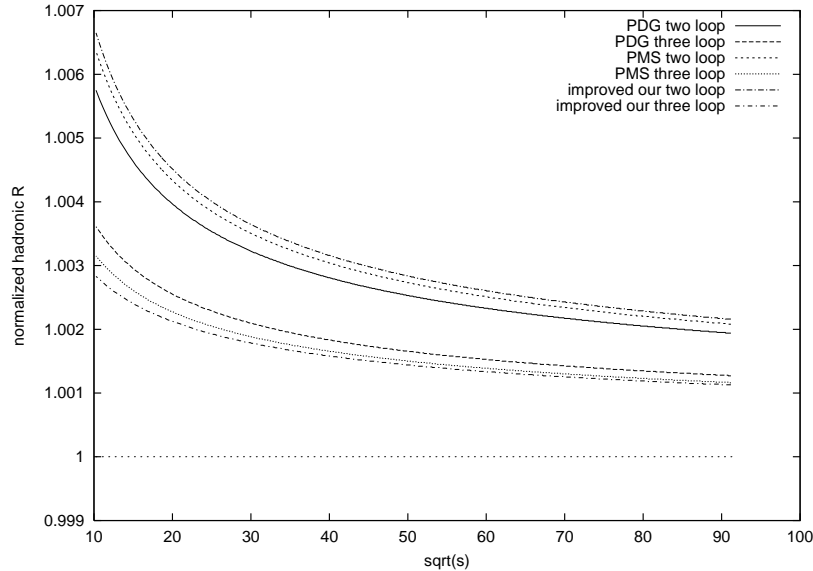


Figure 2: hadronic  $R$  as a function of the center of mass energy for  $N_f = 5$ , as found by various methods, normalized by dividing by the PDG-one-loop result. As opposed to the previous graph, our method now gives very similar results to the PDG-method and the PMS-method. From top to bottom we see graphs for improved-our-two-loop, PMS-two-loop, PDG-two-loop, [gap], PDG-three-loop, PMS-three-loop, improved-our-three-loop.

methods that have been proposed to handle the scheme dependence. The first method is the one used by the Particle Data Group. The PDG uses  $\overline{\text{MS}}$  and sets  $\mu = \sqrt{s_{\text{CM}}}$ . The second method that we consider is the PMS-criterion. Information on this can be obtained from [10]. We also plot the result obtained by our method of re-expanding in  $1/X_1$ . We obtain figure 1. In this figure we did not yet use the symmetry of equations (25) and (28) to remove the logarithms. If we use it, we get figure 2. The normalization that we mentioned is done by dividing by the one-loop result that can be found using the PDG-method. This prevents the graphs from being very close to each others. We conclude that our method gives results basically equal to the two other methods provided that it is improved by the use of the symmetry. “Basically equal” means that the uncertainty that results from ignoring the next order is much larger than the uncertainty that comes from the renormalization scheme dependence.

## 10 Comparison with Other Solutions

Other solutions to the problem of renormalization scheme dependence have been proposed. The one that perhaps is most like ours, is by C.J. Maxwell [6]. In fact, in the case where  $\beta_1 = 0$  the one-loop result of Maxwell is identical to ours. He sums some of the higher order terms along with the ones that come from orders where the full result is known. If an  $n$ -loop calculation has been performed,  $X_n$  is known and all terms that contain  $X_i$  with  $i \leq n$  are to be summed. This has the problem that the definition of the  $X_i$  depends on what

parameters are chosen to parameterize the scheme. Maxwell notes this himself in [6]. As we have seen, the invariants are constants of integration, hence they depend on the boundary chosen. Therefore it does not make much sense to resum these, because it will inevitably lead to arbitrariness. Our approach of expanding in the first invariant avoids this problem.

The PMS criterion, introduced in [10] has the big advantage that, apart from our own method, it is the only one that is really completely scheme and convention independent. It has, however, the disadvantage that it is difficult to apply. Here, the optimum value, in some sense of optimal, of scheme dependent parameters is determined. This involves solving transcendental equations containing integrals, and hence can generally only be done numerically. If one is interested in expressing a physical quantity in, say, the numbers of flavours  $N_f$ , this can only be given as a set of equations. By contrast, we have a series expansion in the parameter  $1/\tilde{X}_1$ . Only one equation needs to be solved to obtain  $\tilde{X}_1$  from  $X_1$ . Such a result can easily be expressed in, for instance,  $N_f$ .

The method of calculating hadronic  $R$  used by the Particle Data Group consists of setting  $\mu$  to a “good” value. This has the disadvantage that one has to pick this “good” value alongside with a “good” renormalization scheme. For hadronic  $R$  the PDG makes  $\mu$  equal to the center of mass energy and uses  $\overline{\text{MS}}$ .

Another idea was put forward in [5]. Here we obtain a differential equation for the physical quantity. We have

$$\frac{dR}{ds_{\text{CM}}} = f(R). \quad (35)$$

As before,  $s_{\text{CM}}$  is the energy squared in the center of mass, not the scale. It turns out that the right-hand side is scheme independent, which should not come as a surprise to the reader of this paper. The method has some disadvantages compared to ours. Firstly, this differential equation still needs to be solved. This could be done by giving an initial value at some reference energy, and then expanding in  $\log(s_{\text{CM}}/s_{\text{ref}})$ . This, of course, goes wrong if the energy of an experiment starts to differ significantly from the reference energy. Furthermore, the mere mention of a “reference energy” indicates that we are reintroducing arbitrariness. So, presumably, we are not to solve this differential equation by expanding with respect to this quantity. What are we to do then? Solving the differential equation numerically, perhaps? The reader will not find it difficult to think of disadvantages of this. Furthermore, we would want to relate different physical quantities to each other. In this method we would give a series expansion that expresses one into the other. However, which of all possible physical quantities is going to appear in a listing of fundamental quantities? And at what energy is this quantity going to be listed? This method has no preference for a particular quantity or energy.

Yet another idea can be found in [3]. Here an equation that looks a lot like equation (2) is given. In our notation it would be given by

$$X_1 = -\frac{r_0^2}{R} + \frac{r_0\beta_1}{\beta_0} \log \frac{r_0\beta_0 + \beta_1 R}{\beta_1 R} + \int_0^R dR' \left( \frac{1}{Y(R')} - \frac{1}{Y_0(R')^2 + Y_1(R')^3} \right), \quad (36)$$

where  $Y(R)$  is a power series that starts with the term  $Y_0(R)^2$ . The  $Y_i$  are invariants related to the  $X_i$  but their definition is not entirely the same. If we go to a scheme where all  $r_i$  with  $i > 0$  are zero, the just given equation

and equation (2) are the same. Because only renormalization scheme invariants occur in equation 36, it must hold in any scheme. This could be an answer to the question that bothered us in the previous paragraph, because this expression is a solution to the differential equation (35). The relation to our method is that we propose to simplify this expression by turning it into an expansion. Numerical accuracy is then achieved by resummation of logarithms.

The method proposed by [4] ultimately boils down to the same equation (36). The philosophy is different though. The idea is that the coupling constant should be interpreted as an “effective charge” that no longer receives higher order corrections.

## 11 Conclusions

We can get rid of the unphysical dependence of physical quantities on the renormalization scheme by expanding in  $1/X_1$  where  $X_1$  is the renormalization scheme invariant that occurs on the one loop level. Numerical accuracy is achieved by resumming all logarithms that contain  $X_1$  into the quantity  $1/\tilde{X}_1$ .

For future research it would be interesting to look at the possibility of generalizing our approach to the case of a theory with masses and/or multiple coupling constants. Another thing that could be done is to try a similar approach to factorization scheme dependence as it arises when one studies deep-inelastic scattering. In the context of the method introduced in [6], both renormalization and factorization scale dependence are discussed in [7].

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